

Long distance tunneling

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Abstract

Quantum tunneling between two potential wells in a magnetic field can be strongly increased when the potential barrier varies in the direction perpendicular to the line connecting the two wells and remains constant along this line. A periodic structure of the wave function is formed in the direction joining the wells. The resulting motion can be coherent like motion in a conventional narrow band periodic structure. A particle penetrates the barrier over a long distance which strongly contrasts to WKB-like tunneling. The whole problem is stationary. The coherent process can be influenced by dissipation.

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Can a particle move under a long and almost classical potential barrier where classical motion is impossible?

According to Wentzel, Kramers, and Brillouin (WKB) [1], there is a finite probability $w \sim \exp(-A)$ of quantum tunneling through an one-dimensional potential barrier. This probability becomes negligible for semiclassical barriers when $A = 2\text{Im}S/\hbar$ and the classical under-barrier action $\text{Im}S$ is big. In two-dimensions the most convenient way to calculate the exponent A is by use of a classical trajectory $x(\tau)$, $y(\tau)$ in imaginary time $t = i\tau$ [2–6]. The trajectory goes in a classically forbidden area (under the barrier) and connects two classically allowed regions having zero velocities at the borders of classical regions. The classical action, constructed by means of this trajectory, is called Euclidean action and determines WKB-type exponent A . The method of classical trajectories in imaginary time is powerful and relatively simple since it allows to determine a tunneling probability in the main (no pre-exponent) approximation $\exp(-A)$ just only solving Newton’s equation of motion.

The problem of quantum tunneling in magnetic field was addressed in Refs. [7–9]. In the Landau gauge there is a parabolic gauge potential $m\omega_c^2(x - x_0)^2/2$ superimposed upon the tunnel barrier potential. The cyclotron frequency is $\omega_c = eH/mc$ and tunneling occurs in the x -direction. If the tunnel barrier is not a constant, containing weak impurity centers, x_0 becomes spatially dependent resulting in a variable gauge potential of a sawtooth shape instead of pure parabolic one. This potential is “pinned” by impurities, separated by the characteristic distance b , and repeats their positions [7,8]. In the regime of a strong magnetic field, when the energy $m\omega_c^2 b^2/2$ exceeds a height of the tunnel barrier, an electron tunnels incoherently through each peak of the gauge potential. For this incoherent motion the total probability of tunneling is a product of partial ones.

This process can be elegantly described in terms of classical trajectory in imaginary time ($t = i\tau$) [10] (see also [11,12]). The classical trajectory is constructed in the following way: $x(\tau)$ is real and corresponds to a translational motion in the direction of tunneling (x -axis) but the transverse coordinate $y = i\eta(\tau)$ is imaginary and performs oscillations in τ . The oscillations are produced by an oscillatory force in the x -direction since impurities are distributed along the x -axis. This potential force is balanced by the x -component of the Lorentz force $m\omega_c \partial\eta/\partial\tau$. The tunnel potential $U(x, y)$ is supposed to be even with respect to y and the classical trajectory satisfies the relation $U[x(\tau), i\eta(\tau)] = E$. At the ends of the trajectory the relation $\eta = 0$ holds which allows to match with the physical $y = 0$.

If the magnetic field is not big, kinetic energy enters the game and a scenario of tunneling can be dramatically different. We consider the two-dimensional tunnel potential which in the barrier region has the form $U(x, y) = u(y)$, tunneling occurs in the x -direction, and the magnetic field is aligned along the z -axis. In this case a variable gauge potential can form under the barrier a certain structure periodic in the direction of tunneling with a period Δx . An electron moves in this periodic potential in the way similar to conventional motion in a periodic structure with a narrow energy band ΔE where tunneling processes through subsequent periodic barriers are strongly coherent. For a conventional periodic structure a wave packet can pass over a long distance with no exponential decrease in amplitude but only with a delay time $\hbar/\Delta E$ as a velocity in a narrow band is proportional to ΔE [13]. In our case, since the potential is x -independent in its barrier part, a period in the x -direction $\Delta x = \sqrt{2|E|/m} \Delta\tau$ ($\sqrt{2|E|/m}$ plays a role of velocity) has an intrinsic nature. As shown

below, the time $\Delta\tau$ is a period of oscillations in the well associated with the potential

$$v(\eta) = u(i\eta) - \frac{m\omega_c^2}{2} (\eta + \eta_0)^2 \quad (1)$$

The particle energy E is negative, $\eta_0 = \sqrt{2|E|/m\omega_c^2}$, and the minimum of $u(y)$ corresponds to $u(0) = 0$. The transverse oscillatory motion in the direction of $y = i\eta$ is coupled to the translational motion (x -direction) due to the Lorentz force.

When the magnetic field is close to the certain value H_R the probability of tunneling $w \sim \exp(-A)$ through a long barrier becomes not exponentially small, like for a conventional narrow band dynamics, and corresponds to $A \rightarrow 0$. This is a situation of Euclidean resonance studied in papers [14–18] for tunneling through nonstationary barriers when also $A \rightarrow 0$ at a certain value of an ac amplitude. Therefore, a phenomenon of Euclidean resonance has rather a general nature since it occurs also in a static barrier. As argued in this paper, the magnetic field sets a long distance under-barrier coherence which allows under-barrier motion over a long distance. This strongly contrasts to WKB-like tunneling.

Below we formulate the above arguments in terms of a classical trajectory in imaginary time ($t = i\tau$). A potential barrier, part of which is plotted in Fig. 1, is even with respect to y , it does not depend on x at $0 < x < R$ where $U(x, y) = u(y)$ ($u(0) = 0$), and the function $U(x, 0)$ has jumps at $x = 0$ and $x = R$. Tunneling occurs between two classically allowed regions $x < 0$ and $R < x$. For convenience, the potential in Fig. 1 is drawn in a way that it is a constant at $x < 0$ and $R < x$. This condition is not necessary and $U(x, y)$ can correspond to tunneling between two quantum wires or two quantum dots. In order to calculate a tunneling probability in the exponential approximation one can know only a classical trajectory in imaginary time connecting two points $\{x = 0, y = 0\}$ at the moment $\tau = \tau_0$ and $\{x = R, y = 0\}$ at the moment $\tau = 0$. Classical equation of motion under the barrier have the form

$$m \frac{\partial^2 x}{\partial \tau^2} + m\omega_c \frac{\partial \eta}{\partial \tau} = 0; \quad m \frac{\partial^2 \eta}{\partial \tau^2} + m\omega_c \frac{\partial x}{\partial \tau} + \frac{\partial u(i\eta)}{\partial \eta} = 0 \quad (2)$$

The total energy conserves

$$E = -\frac{m}{2} \left(\frac{\partial x}{\partial \tau} \right)^2 + \frac{m}{2} \left(\frac{\partial \eta}{\partial \tau} \right)^2 + u(i\eta) \quad (3)$$

The conditions to Eqs. (2) are

$$\left. \frac{\partial \eta}{\partial \tau} \right|_0 = \left. \frac{\partial \eta}{\partial \tau} \right|_{\tau_0} = 0; \quad \eta(0) = \eta(\tau_0) = 0 \quad (4)$$

According to Eq. (3),

$$\left. \frac{\partial x}{\partial \tau} \right|_0 = \left. \frac{\partial x}{\partial \tau} \right|_{\tau_0} = -\sqrt{\frac{2|E|}{m}} \quad (5)$$

For a smooth potential all velocities should be zero at the ends of a trajectory but in our case $\partial x/\partial \tau$ is finite due to jumps in the potential energy at $x = 0$ and $x = R$. The differential

equations (2), with respect to the functions $\partial x/\partial\tau$ and η , depend on three parameters which should be determined from the six conditions (4) and (5). Since the functions $\partial x/\partial\tau$ and η are periodic among the six conditions there are only three independent ones.

The probability of tunneling $w \sim \exp(-A)$ from $x = 0$ to $x = R$ in Fig. 1 is expressed through the Euclidean action [2,3,5,6,10–12,19–21]

$$A = \frac{2}{\hbar} \int_0^{\tau_0} d\tau \left[\frac{m}{2} \left(\frac{\partial x}{\partial \tau} \right)^2 - \frac{m}{2} \left(\frac{\partial \eta}{\partial \tau} \right)^2 + m\omega_c \eta \frac{\partial x}{\partial \tau} + u(i\eta) - E \right] \quad (6)$$

Without a magnetic field $\eta = 0$ and the action (6) coincides with the WKB expression. With the solution of the first equation (2), $\partial x/\partial\tau = -\omega_c(\eta + \eta_0)$, the action (6) reads

$$A = \frac{2}{\hbar} \int_0^{\tau_0} d\tau \left[-\frac{m}{2} \left(\frac{\partial \eta}{\partial \tau} \right)^2 + v(\eta) - E - m\omega_c \eta_0 \frac{\partial x}{\partial \tau} \right] \quad (7)$$

By means of Eq. (3) the expression (7) takes the form

$$A = A_{WKB} - \frac{4}{\hbar} \int_0^{\tau_0} d\tau [E - v(\eta)] \quad (8)$$

where $A_{WKB} = 2\sqrt{2m|E|}R/\hbar$ comes from the last term in Eq. (7) and it is the WKB action related to one-dimensional tunneling across a rectangular barrier of the length R with the energy $|E|$ below the barrier top. A_{WKB} in Eq. (8) is generic with the conventional under-barrier action in a multi-dimensional case [2,3,5,6]. The second (negative) term in Eq. (8) is solely due to the magnetic field and corresponds to the transverse motion. The total energy now has the form

$$E = \frac{m}{2} \left(\frac{\partial \eta}{\partial \tau} \right)^2 + v(\eta) \quad (9)$$

In contrast to the x -motion, kinetic energy in the transverse channel does not change sign compared to the physical trajectory $(\partial y/\partial t)^2 = (\partial \eta/\partial \tau)^2$. Therefore, the transverse motion occurs at the region where $v(\eta) < E$ and the second term in Eq. (8) is negative. With the expression (9) one can describe an oscillatory motion in the potential $v(\eta)$. We specify a shape of $v(\eta)$ as in Fig. 2 where $v(\Delta\eta) = E$. The trajectory $\eta(\tau)$ is drawn in Fig. 3(a) where the period $\Delta\tau$, according to Eq. (9), is

$$\Delta\tau = \sqrt{2m} \int_0^{\Delta\eta} \frac{d\eta}{\sqrt{E - v(\eta)}} \quad (10)$$

The trajectory $x(\tau)$ is shown in Fig. 3(b). Each cycle of $\eta(\tau)$ in the potential well in Fig. 2 results in the translation of $x(\tau)$ by Δx determined by

$$\Delta x = \omega_c \sqrt{2m} \int_0^{\Delta\eta} d\eta \frac{\eta_0 + \eta}{\sqrt{E - v(\eta)}} \quad (11)$$

The trajectory in Fig. 3 looks qualitatively similar as one related to tunneling through a barrier slightly violated by impurities in a strong magnetic field [10,11]. The difference is

that the oscillations in Fig. 3 have an intrinsic nature but in [10,11] they are determined by distributed impurities.

This method of trajectories is applicable when the distance between the wells is $R = N\Delta x$ (and also $\tau_0 = N\Delta\tau$), since at the ends ($x = 0$ and $x = R$) the condition $\eta = 0$ should hold. Fig. 3 is plotted for $N = 3$. By means of (9), the last term in Eq. (8) can be written as $(-N\Delta A)$ where

$$\Delta A = \frac{4\sqrt{2m}}{\hbar} \int_0^{\Delta\eta} d\eta \sqrt{E - v(\eta)} \quad (12)$$

and the action (8) takes the form

$$A = A_{WKB} - N\Delta A = \left(\frac{2\sqrt{2m|E|}}{\hbar} - \frac{\Delta A}{\Delta x} \right) R \quad (13)$$

At a small magnetic field $E - v(i\eta) \simeq -u(i\eta)$ and $\Delta x \sim \ln(1/H)$ is logarithmically big. Therefore, at $H \rightarrow 0$ the action (13) turns to its conventional limit A_{WKB} .

The action (13) and its WKB part are shown in Fig. 4(a) at the points $R = N\Delta x$ where they have only sense. If R is not $N\Delta x$ another method of calculation is required. The action A is substantially reduced compared to its WKB part A_{WKB} . When the magnetic field is close to the certain value H_R the action (13) is $A \sim (H_R - H)$ and may turn to zero. This means that the tunneling probability $w(H)$ becomes not exponentially small at $H = H_R$. As mentioned above, the phenomenon, when $A \rightarrow 0$, is called Euclidean resonance.

The method of trajectory used corresponds to one-instanton approach and it holds at $H < H_R$ when $\exp(-A)$ is small. At $H > H_R$ the exponent $\exp(-A)$ is not small and one should apply a multi-instanton approach which accounts all powers of the exponent. This is a matter of a further study. One has to expect that at $H > H_R$ the tunneling probability $w(H)$ also decays in a manner as at $H < H_R$ having the peak at $H = H_R$ plotted in Fig. 5. The peak width, as one can show, is roughly H_R/A_{WKB} . The condition $R = N\Delta x(H)$, when the above method of trajectories is applicable, holds at certain values of the magnetic field $H = h_N$ shown in Fig. 5. The relation

$$w(h_{N-1}) = w(h_N) \exp(-\Delta A) \quad (14)$$

says that always exists some h_N for which $w(h_N)$ is not smaller than $\exp(-\Delta A)$.

The oscillatory structure of the trajectory in the x -direction indicates analogous oscillations in the wave function. At $H < H_R$ and $R = N\Delta x$ the function $|\psi(x, 0)|^2$ is drawn in Fig. 4(b) where $|\psi(N\Delta x, 0)/\psi(0, 0)|^2 \sim \exp(-A)$. Fig. 4(b) is correct only when the terminal point is $R = N\Delta x$. Fig. 4(b) does not serve for $H \neq h_N$ by a simple shift of the terminal point away from $N\Delta x$.

Under the condition of Euclidean resonance $H = H_R$ the function $|\psi(x, 0)|^2$ in Fig. 6 is periodic at least with the exponential accuracy provided by the method of trajectories. This means that $|\psi(x, 0)|^2$, besides the periodic oscillations, may have a power law decay. The spatial oscillations and not exponentially small tunneling through a long barrier draw the analogy with motion in a conventional narrow band periodic structure. Like this motion, in our case there is a long distance under-barrier coherence set by the magnetic field. One can interpret this as formation of a periodic variable gauge potential.

An interference of various paths in the problem of localization in disordered systems [22] does not depend on a particular shape of an impurity potential and is determined by a mean free path. In contrast to this, the under-barrier coherence strongly depends on a potential shape since $v(\eta)$ should have a well as in Fig. 2. This rule provides a choice of $u(y)$. An arbitrary $u(y)$, for example $u(y) = u_0 y^2/a^2$, does not result in that well but the potential $u(y) = u_0 (y^2/a^2 + y^4/a^4)$ does. Analogously, a pure harmonic potential $u(y) = u_0 (1 - \cos y/a)$ is not suitable and should be supplemented by a double harmonic at least.

Below we consider, as an example, the double harmonic potential $u(y) = u_0 (1 - \cos y/a) (1 - \lambda \cos y/a)$ which is created between two quantum dots on a surface with a perpendicular magnetic field. It is sufficient to have this analytical form only at the length $|y| \lesssim \Delta\eta$. The dots are separated by the distance R . Sizes of the dots determine the distance between their discrete energy levels which is supposed to be $|E| \simeq 0.01$ eV [23]. The parameters of the double harmonic potential are $a = 50$ Å, $\lambda = 0.215$, and $u_0 = 1$ eV. The spatial period of the potential is $2\pi a = 314$ Å. After calculations one can obtain $H_R \simeq 10$ Tesla, $\Delta x \simeq 120$ Å, $\Delta\eta \simeq 110$ Å, and $\Delta A \simeq 12$. At $H = H_R$ the tunneling probability does not fall exponentially with R which enables to consider tunneling through long barriers, say, $R = 1$ cm.

The long distance under-barrier coherence can be influenced by dissipation. Dissipation results in a finite width δE of energy levels inside the wells and also disturbs an under-barrier motion, according to Caldeira and Leggett [24]. In the classical dynamics dissipation corresponds to the form $m\ddot{x} + m\gamma\dot{x}$. Using the theory [24], one can obtain (we omit details) the criterion $0.2\gamma\tau_0 < 1$ when dissipation does not influence the frictionless motion. Since $\delta E \sim \hbar\gamma$ this criterion is equivalent to $\delta E/E < 7.1\lambda/R$ where the parameter $\lambda = \hbar/\sqrt{m|E|}$ can be interpreted as a de Broglie wave length. When $\delta E/E \sim 0.1$ [23] one can estimate $R < 1000$ Å. For a bigger R dissipation modifies results and this is a matter of a further study. A non-homogeneity $\delta u(x)$ (including applied voltage) of a barrier in the x -direction does not violate the above results as soon as $\delta u(x) < 4|E|$.

In summary, an answer to the question in the beginning of the paper is positive. The nature allows a long distance under-barrier motion which is counterintuitive and contrasts to WKB-like tunneling. An example of this motion, tunneling between two wells in a magnetic field, is considered in the paper. In the absence of dissipation the magnetic field sets a long distance coherence under the barrier and a particle can tunnel through a long potential barrier as through a conventional narrow band periodic structure. This phenomenon relates to Euclidean resonance.

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FIGURES

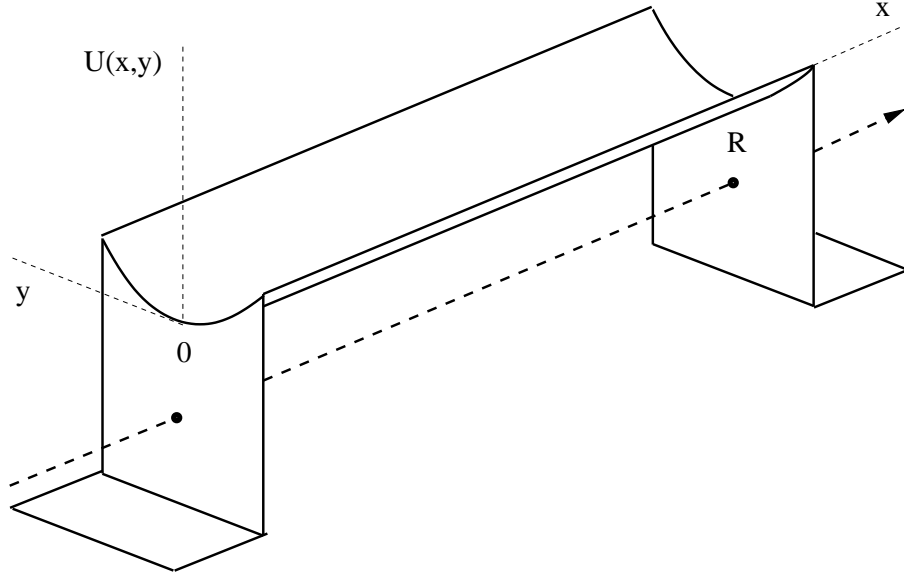


FIG. 1. A particle passes (the dashed line) through the tunnel potential.

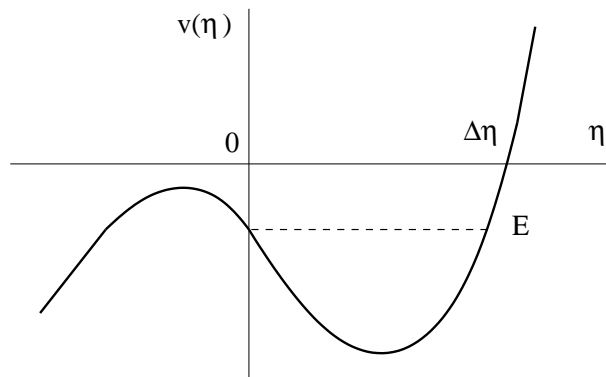


FIG. 2. The effective potential for the transverse motion forms the well.

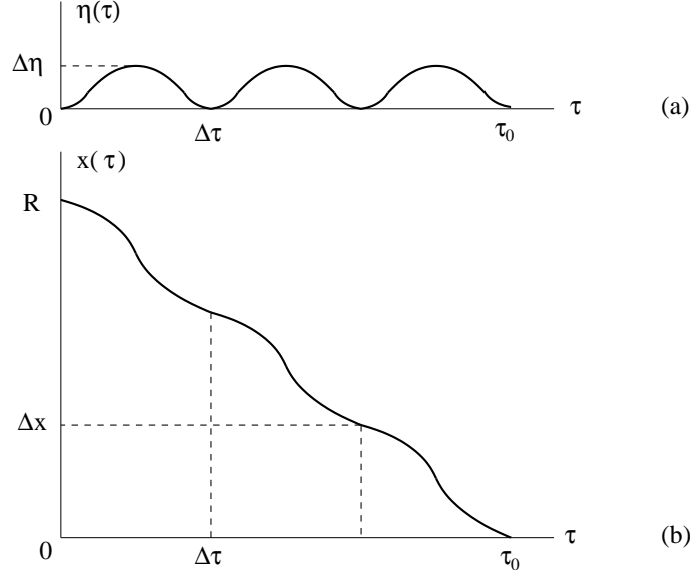


FIG. 3. The classical trajectory in imaginary time. It is chosen $N = 3$. (a) The transverse component ($y = i\eta$). (b) The motion in the direction of tunneling.

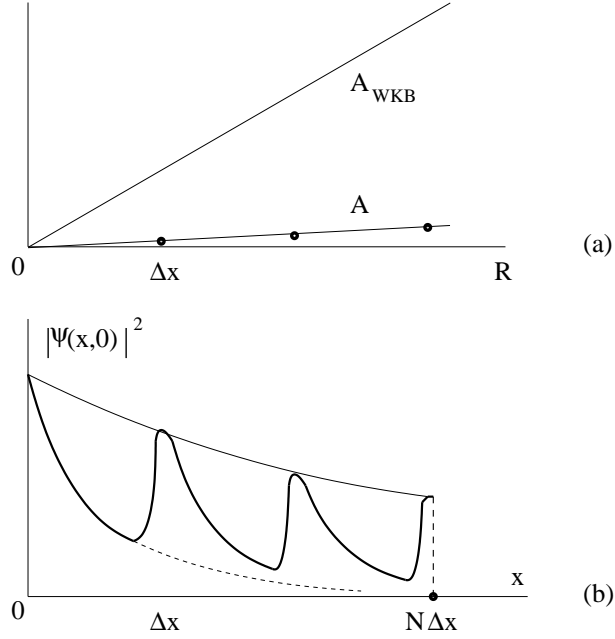


FIG. 4. $H < H_R$; (a) The action A and its WKB part as a function of distance $R = N\Delta x$, shown by dots, between the two wells. (b) Oscillations in the wave function when $R = N\Delta x$ ($N = 3$). The dashed curve shows the WKB-like dependence in the absence of the magnetic field.

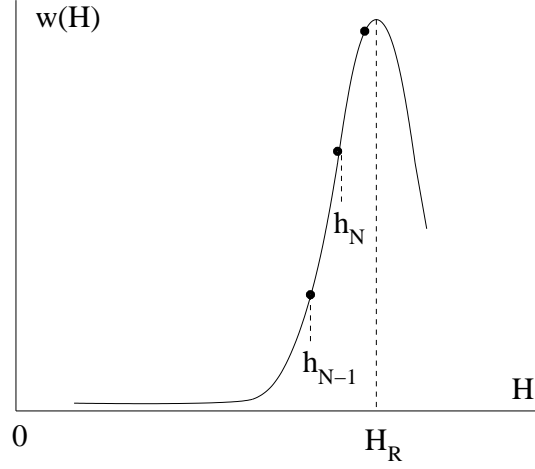


FIG. 5. The probability of tunneling as a function of magnetic field has the peak at H_R (Euclidean resonance). h_N correspond to the condition $R = N\Delta x$.

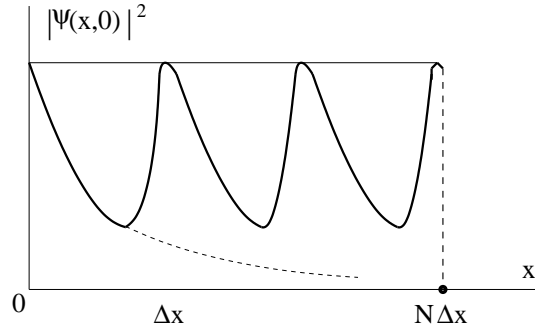


FIG. 6. The case of Euclidean resonance $H = H_R$. The total distance is $R = N\Delta x$ ($N = 3$). The dashed curve represents the WKB-like dependence.